# TJ USAMO Practice 3

VMT Math Team

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1. (USAMO 1979) Determine all non-negative integral solutions $(n_1, n_2, \ldots, n_{14})$ if any, apart from permutations, of the Diophantine equation

$$n_1^4 + n_2^4 + \cdots + n_{14}^4 = 1,599$$

2. Prove that, for $a, b, c > 0$,

$$a^3 + b^3 \geq a^2 b + ab^2$$

and

$$\frac{1+a^2}{b} + \frac{1+b^2}{c} + \frac{1+c^2}{a} \geq 6$$

3. (MOP 2003) Let $ABC$ be a triangle, and let $H$ and $\omega$ be its orthocenter and circumcircle respectively. Let $BD$ be a diameter of $\omega$. Prove that $AHCD$ is a parallelogram.

4. (MOP 2003) Find

$$\sum_{k=1}^{n} k!(k^2 + k + 1)$$

in closed form.

5. (USAMO 1996) An ordered $n$-tuple

$$(x_1,\, x_2, \ldots,\, x_n)$$

in which each term is either 0 or 1 is called a *binary sequence of length n*. Let $a_n$ be the number of binary sequences of length $n$ containing no three consecutive terms equal to 0, 1, 0 in that order. Let $b_n$ be the number of binary sequences of length $n$ that contain no four consecutive terms equal to 0, 0, 1, 1 or 1, 1, 0, 0 in that order. Prove that $b_{n+1} = a_n$ for all positive integers $n$.