TJ USAMO Practice 4 Solutions

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1. Show that if there are n people at a party, then two of them know the same number of people (knowing is reciprocal).

   Solution

   We will prove by contradiction that some two people must know the same number of people.

   We assume otherwise - that no two people know the same number or people. There are exactly n distinct people, so any one person can know $0, 1, 2, 3, \ldots,$ or $n-1$ people. Because one person can know one of the n numbers of people, by the pidgeonhole principle, exactly one person knows each number of people. Then there is a person P who knows 0 other people, and there is a person Q who knows $n-1$ people. Q knows $n-1$ people, so Q knows everybody else, so that person knows P, so P in turn must know Q because knowing somebody is reciprocal. This, however, is impossible if P doesn't know anybody. Thus, our assumption must be false.

2. Find, with proof, the minimum value of

$$\frac{a^3}{4b} + \frac{b}{8c^2} + \frac{1+c}{2a}$$

   where a, b, and c are positive real numbers, and determine all values of a, b and c where this value is obtained.

   Solution

   We break up the third term and apply AM-GM to show that the minimum is at least $\frac{5}{4}$:

$$\frac{a^3}{4b} + \frac{b}{8c^2} + \frac{1+c}{2a} = \frac{a^3}{4b} + \frac{b}{8c^2} + \frac{1}{2a} + \frac{c}{4a} + \frac{c}{4a} \geq 5\sqrt[5]{\frac{a^3}{4b}\frac{b}{8c^2}\frac{1}{2a}\frac{c}{4a}\frac{c}{4a}} = \frac{5}{4}$$

   The equality condition for AM-GM is that all of the terms are equal, which means that for $\frac{a^3}{4b} + \frac{b}{8c^2} + \frac{1+c}{2a}$ to be $\frac{5}{4}$, $\frac{1}{4} = \frac{1}{2a} = \frac{a^3}{4b} = \frac{c}{4a} = \frac{b}{8c^2}$. We solve this set of equations to find $(a, b, c) = (2, 8, 2)$, and verify that the minimum is $\frac{5}{4}$.

3. (USAMO) Let $ABCD$ be a convex quadrilateral whose diagonals are orthogonal, and let P be the intersection of the diagonals. Prove that the four points that are symmetric to P with respect to the sides form a cyclic quadrilateral.

   Solution

   Let P_1, P_2, P_3, and P_4 be the reflections of P over $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{DA}$ respectively. Let P_i' be the midpoint of $\overline{PP_i}$, where $i = 1, 2, 3, 4$. The P_i' are on $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{DA}$ respectively, and are projections of P onto sides of $ABCD$. Thus, because they are dialations of each other from point P, $P_1P_2P_3P_4$ is cyclic $\iff$ $P_1'P_2'P_3'P_4'$ is cyclic.

   Because $\angle PP_1'B$ and $\angle PP_2'B$ are both right angles, $PP_1'BP_2'$ is a cyclic quadrilateral $\implies m\angle P_1'PB = m\angle P_1'P_2'B = \alpha \implies m\angle PP_2'P_1' = 90^o - \alpha$. Similarly, $m\angle APP_1' = \beta \implies m\angle P_1'P_4'P = 90^o - \beta$. Because the diagonals of $ABCD$ are perpendicular, $\alpha + \beta = 90^o \iff m\angle P_1'P_4'P + m\angle PP_2'P_1' = 180^o - \alpha - \beta = 90^o$. Similarly, $m\angle PP_4'P_3' + m\angle P_3'P_2'P = 90^o \implies m\angle P_1'P_4'P_3' + m\angle P_3'P_2'P_1' = 180^o \iff P_1'P_2'P_3'P_4'$ is cyclic $\iff$ $P_1P_2P_3P_4$ is cyclic. *Q.E.D.*

4. A sequence x_n of positive reals satisfies $x_{n-1}x_{n+1} \leq x_n^2$. Let a_n be the average of the terms $x_0, x_1, ..., x_n$ and b_n be the average of the terms $x_1, x_2, ..., x_n$. Show that $a_nb_{n-1} \geq a_{n-1}b_n$.

   Solution

   Let $S_n = x_1 + x_2 + \cdots + x_n$.

   We want to show that: $\frac{(x_0+x_1+\cdots+x_n)}{n+1} \cdot \frac{(x_1+x_2+\cdots+x_{n-1})}{n-1} \geq \frac{(x_0+x_1+\cdots+x_{n-1})}{n} \cdot \frac{(x_1+x_2+\cdots+x_n)}{n}$

   $\iff n^2(S_{n-1} + x_0 + x_n)(S_{n-1}) \geq (n^2 - 1)(S_{n-1} + x_0)(S_{n-1} + x_n)$

   $\iff (S_{n-1})^2 + x_0S_{n-1} + x_nS_{n-1} + x_0x_n \geq n^2(x_0x_n)$

   $\iff (S_{n-1} + x_0)(S_{n-1} + x_n) \geq n^2(x_0x_n)$

   $\iff (x_0 + \cdots + x_{n-1})(x_1 + \cdots + x_n) = (x_0 + \cdots + x_{n-1})(x_n + \cdots + x_1) \geq n^2(x_0x_n)$

   $\Longleftarrow^1 (\sqrt{x_0x_n} + \sqrt{x_1x_{n-1}} + \cdots + \sqrt{x_{n-1}x_1})^2 \geq n^2(x_0x_n)$

   All we need to show is the last inequality. To do this, we first prove a lemma: $x_mx_n \geq x_{m-i}x_{n+i} \quad \forall m, n, i \in \mathbb{Z}^+ : i \leq m \leq n$. $\prod_{i=m}^{n}(x_i)^2 \geq \prod_{i=m}^{n}(x_{i-1}x_{i+1}) \iff (x_mx_{m+1}\cdots x_n)^2 \geq x_{m-1}x_m(x_{m+1}x_{m+2}\cdots x_{n-1})^2x_nx_{n+1} \iff x_mx_n \geq x_{m-1}x_{n+1}$, and this identity, when applied for i iterations, yields the lemma. With this fact we return to the inequality:

[1] This follows from the Cauchy Inequality.

$$(\sqrt{x_0x_n}+\sqrt{x_1x_{n-1}}+\cdots+\sqrt{x_{n-1}x_1})^2 \geq (\sqrt{x_0x_n}+\sqrt{x_0x_n}+\cdots+\sqrt{x_0x_n})^2 = n^2(x_0x_n),$$

from which the desired result follows.