TJ USAMO Practice 4

VMT Math Team

November 17, 2003

1. Show that if there are n people at a party, then two of them know the same number of people (knowing is reciprocal).

2. Find, with proof, the minimum value of

$$\frac{a^3}{4b} + \frac{b}{8c^2} + \frac{1+c}{2a}$$

where a, b, and c are positive real numbers, and determine all values of a, b and c where this value is obtained.

3. (USAMO) Let $ABCD$ be a convex quadrilateral whose diagonals are orthogonal, and let P be the intersection of the diagonals. Prove that the four points that are symmetric to P with respect to the sides form a cyclic quadrilateral.

4. A sequence x_n of positive reals satisfies $x_{n-1}x_{n+1} \leq x_n^2$. Let a_n be the average of the terms $x_0, x_1, ..., x_n$ and b_n be the average of the terms $x_1, x_2, ..., x_n$. Show that $a_n b_{n-1} \geq a_{n-1} b_n$.