TJ USAMO Practice 5 Solutions

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1. Prove that for all $a, b, c > 0$

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}$$

Solution

We verify this inequality by an application of AM-HM[1]

$$\begin{aligned} \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2} \iff \frac{a+b+c}{b+c} + \frac{a+b+c}{c+a} + \frac{a+b+c}{a+b} &\geq \frac{9}{2} \\ \iff \frac{\frac{a+b+c}{b+c} + \frac{a+b+c}{c+a} + \frac{a+b+c}{a+b}}{3} &\geq \frac{3}{2} \\ \Longleftarrow \frac{3}{\frac{b+c}{a+b+c} + \frac{c+a}{a+b+c} + \frac{a+b}{a+b+c}} &\geq \frac{3}{2} \\ \iff \frac{3(a+b+c)}{2(a+b+c)} &\geq \frac{3}{2} \end{aligned}$$

The last line is clearly true. Q.E.D.

2. (MOP 1998) Prove that for $x, y, z > 0$

$$\frac{x}{(x+y)(x+z)} + \frac{y}{(y+z)(y+x)} + \frac{z}{(z+x)(z+y)} \leq \frac{9}{4(x+y+z)}$$

Solution

[1] If it is ever possible to manipulate an inequality such that a sum consists of terms with the same numerator, an application of AM-HM will usually work nicely, as in this case. Because a, b, and c each appear once in every fraction, we can easily manipulate the numerators to be identical.

We eliminate the denominators by multiplying through by $4(x+y)(x+z)(y+z)(x+y+z)$.[2] After cancellation, the remainder is AM-GM:

$$
\begin{array}{ll}
& \dfrac{x}{(x+y)(x+z)} + \dfrac{y}{(y+z)(y+x)} + \dfrac{z}{(z+x)(z+y)} \leq \dfrac{9}{4(x+y+z)} \\
\Longleftrightarrow & 4(x+y+z)(x(y+z)+y(x+z)+z(x+y)) \leq 9(x+y)(y+z)(z+x) \\
\Longleftrightarrow & 4((x^2y+x^2z+x^2y+xyz+x^2z+xyz)+(xy^2+xyz+xy^2+y^2z+xyz+y^2z) \\
& +(xyz+xz^2+xyz+yz^2+xz^2+yz^2)) \leq 9(x^2y+x^2z+xy^2+y^2z+xz^2+yz^2+2xyz) \\
\Longleftrightarrow & 8(\sum_{Sym} x^2y)+24xyz \leq 9(\sum_{Sym} x^2y)+18xyz \\
\Longleftrightarrow & 6xyz \leq x^2y+x^2z+xy^2+y^2z+xz^2+yz^2
\end{array}
$$

The last line is easily shown by AM-GM, and we are done.

3. Prove that for positive real numbers a, b, c with $a+b+c=1$,

$$a^3+b^3+c^3 \geq \frac{1}{9}$$

Solution

We apply the Power-Mean inequality for $r=3$ and $r'=1$ to obtain:

$$
\begin{array}{rcl}
a^3+b^3+c^3 & \geq & \dfrac{1}{9} \\
\Longleftrightarrow (\dfrac{a^3+b^3+c^3}{3})^1 & \geq & \dfrac{1}{27} \\
\Longleftarrow (\dfrac{a^1+b^1+c^1}{3})^3 & \geq & \dfrac{1}{27} \\
\Longleftrightarrow \dfrac{a+b+c}{3} & \geq & \dfrac{1}{3}
\end{array}
$$

By the given $a+b+c=1$, the last line is true. Q.E.D.

4. (MOP 2003) Prove that for all nonzero $a, b, c \in \mathbb{R}$,

$$\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2} \geq \frac{a}{c}+\frac{c}{b}+\frac{b}{a}$$

[2] When considering multiplying an inequality out, check to see how many distinct terms you will have to multiply by. Although there are 8 polynomials in the denominators, there are a lot of duplicates and most are binomials, so this will not be that bad...

Solution

We note that the LHS is positive, and that $|x| \geq x \quad \forall x \in \mathbb{R}$. Cauchy yields,

$$\begin{aligned}
(\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2})^2 &= (\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2})(\frac{b^2}{c^2}+\frac{c^2}{a^2}+\frac{a^2}{b^2}) \geq (\frac{a}{c}+\frac{c}{b}+\frac{b}{a})^2 \\
\iff \frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2} &= |\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}| \geq |\frac{a}{c}+\frac{c}{b}+\frac{b}{a}| \geq \frac{a}{c}+\frac{c}{b}+\frac{b}{a}
\end{aligned}$$

Q.E.D.

5. (Titu97) Prove that for positive $x_1, x_2, \ldots, x_n$,

$$\frac{x_1^3}{x_1^2+x_1x_2+x_2^2}+\frac{x_2^3}{x_2^2+x_2x_3+x_3^2}+\cdots+\frac{x_n^3}{x_n^2+x_nx_1+x_1^2} \geq \frac{x_1+x_2+\cdots+x_n}{3}$$

Solution

We first prove a lemma. Lemma: $\frac{\sum_{i=1}^n x_i}{2} \leq \sum_{Cyc} \frac{x_1^3}{x_1^2+x_2^2}$. The lemma follows from the following algebra:

$$\begin{aligned}
\sum_{Cyc} \frac{x_1}{2} &= \sum_{Cyc} \frac{x_1x_2^2}{2x_1x_2} \geq \sum_{Cyc} \frac{x_1x_2^2}{x_1^2+x_2^2} = \sum_{Cyc} x_1(\frac{x_2^2+x_1^2-x_1^2}{x_1^2+x_2^2}) \\
&= \sum_{Cyc} x_1(1-\frac{x_1^2}{x_1^2+x_2^2}) = (\sum_{i=1}^n x_i) - \sum_{Cyc} \frac{x_1^3}{x_1^2+x_2^2} \\
\iff \frac{\sum_{i=1}^n x_i}{2} &\geq (\sum_{i=1}^n x_i) - \sum_{Cyc} \frac{x_1^3}{x_1^2+x_2^2} \\
\iff \sum_{Cyc} \frac{x_1^3}{x_1^2+x_2^2} &\geq \frac{\sum_{i=1}^n x_i}{2}
\end{aligned}$$

Thus,

$$\frac{\sum_{i=1}^n x_i}{3} \leq \sum_{Cyc} \frac{x_1^3}{\frac{3}{2}(x_1^2+x_2^2)} = \sum_{Cyc} \frac{x_1^3}{x_1^2+\frac{x_1^2+x_2^2}{2}+x_2^2} \leq \sum_{Cyc} \frac{x_1^3}{x_1^2+x_1x_2+x_2^2}$$

Q.E.D.