

TJ USAMO Practice 6 Solutions

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1. Let $\triangle ABC$ be equilateral, and let D and E be points on \overline{AB} and \overline{AC} , respectively, such that $AD = CE$. Let \overline{BE} and \overline{CD} meet at F . If $[ABC] = 7$ and $[BCF] = 2$, compute $\frac{BD}{DA}$.

Solution

We extend \overline{AF} past F to G , a point on \overline{BC} . Because $[BCF] = 2$ and $[ABC] = 7$, we must have $\frac{AF}{FG} = \frac{5}{2}$. Because $\triangle ABC$ is equilateral, $AD + DB = AB = AC = AE + EC$. Combined with $AD = CE \implies DB = AE \implies \frac{BD}{DA} = \frac{AE}{EC} = x$. By Ceva's Theorem, we must have $\frac{BG}{GC} = x^2$.

We consider vectors $\vec{b} = \vec{AB}$ and $\vec{c} = \vec{AC}$. By the length ratios above, we have $\vec{AD} = \frac{1}{x+1}\vec{b}$, $\vec{AE} = \frac{x}{x+1}\vec{c}$, and $\vec{AG} = \frac{x^2\vec{b} + \vec{c}}{x^2+1}$. All of the points $P \in \overline{BE}$ can be defined in terms of \vec{b} , \vec{AE} , and a weight $\omega_1 \in [0, 1]$ as $\vec{AP} = \omega_1\vec{b} + (1 - \omega_1)\vec{AE}$. Similarly, all of the points $Q \in \overline{DC}$ can be defined by $\vec{AQ} = \omega_2\vec{AD} + (1 - \omega_2)\vec{c}$. Solving simultaneously produces the solution $\vec{AF} = \frac{x^2+1}{x^2+x+1}\vec{AG}$, from which it follows that $\frac{x^2+1}{x^2+x+1} = \frac{5}{7} \implies x = 2$ or $\frac{1}{2}$.

2. In $\triangle ABC$, $\angle B = 2\angle C$. Prove that $AC^2 = AB^2 + AB \cdot BC$.

Solution

We construct angle bisector \overline{BD} with $D \in \overline{AC}$ which yields $\angle ABD \cong \angle DBC \cong \angle BCA \implies \triangle BDC$ is isosceles and $\frac{AD}{AB} = \frac{DC}{CB} \equiv AD \cdot BC = DC \cdot AB$. Because $\triangle BDC$ is isosceles, $BD = DC$.

Now we apply Stewart's Theorem to triangle ABC , obtaining

$$\begin{aligned} AB^2 \cdot DC + BC^2 \cdot AD &= BD^2 \cdot AC + AD \cdot AC \cdot DC \\ \iff AB^2 \cdot DC + DC \cdot BC \cdot AB &= DC^2 \cdot AC + AD \cdot AC \cdot DC \\ \implies AB^2 + BC \cdot AB &= AC \cdot (DC + AD) = AC^2 \end{aligned}$$

Q.E.D.

3. Let I be the incenter of $\triangle ABC$, and A' the midpoint of $\overset{\vee}{BC}$ of the circumcircle of $\triangle ABC$. Prove that $A'B = A'C = A'I$.

Solution

Because A' is the midpoint of $\overset{\vee}{BC}$, it is clear that $A'B = A'C$, thus we need only to show that $A'B = A'I$.

Because I is the incenter of $\triangle ABC$, \overline{AI} bisects $\angle BAC \iff A, I, A'$ are all collinear. We let $\alpha = m\angle BAC, \beta = m\angle CBA, \gamma = m\angle ACB$, with $\alpha + \beta + \gamma = \pi$. We have $m\angle IAC = \frac{\alpha}{2}$ and $m\angle CBI = \frac{\beta}{2}$. Because $ACA'B$ is a cyclic quadrilateral, $\angle ACB \cong \angle AA'B \implies m\angle AA'B = \gamma$ and $\angle A'BC \cong \angle A'AC \implies m\angle A'BC = \frac{\alpha}{2} \implies m\angle A'BI = \frac{\alpha+\beta}{2}$. Finally, $m\angle BIA' = \pi - m\angle IA'B - m\angle A'BI = \pi - \gamma - \frac{\alpha+\beta}{2} = (\alpha + \beta + \gamma) - \gamma - \frac{\alpha+\beta}{2} = \frac{\alpha+\beta}{2} = m\angle A'BI \implies \triangle A'BI$ is isosceles with $A'B = A'I$. Q.E.D.

4. Let D, E , and F be the feet of the altitudes from A, B , and C , respectively, in $\triangle ABC$. If H is the orthocenter of $\triangle ABC$, then prove that $\triangle AFE, \triangle BDF$, and $\triangle CDE$ are all similar to $\triangle ABC$, and that H is the incenter of $\triangle DEF$.

Solution

Because D and E are the feet of the altitudes from A and B , we have $\angle AEB \cong \angle ADB \iff AEDB$ is cyclic $\iff m\angle ABD + m\angle DEA = \pi \implies \angle ABC \cong \angle ABE \cong \angle CED$. Similarly, $\angle CAB \cong \angle EDC \implies \triangle CED \sim \triangle CBA$. By similar reasoning, $\triangle BDF \sim \triangle BAC$ and $\triangle AEF \sim \triangle ABC$.

Let $m\angle BAC = \alpha$. Because $\angle AEB$ is right, $m\angle EBA = \frac{\pi}{2} - \alpha$. Because $\angle HFB$ and $\angle BDH$ are both right angles, $BDHF$ is cyclic $\iff \angle HBF \cong \angle HDF$, so $m\angle ADF = \frac{\pi}{2} - \alpha$. Similarly, $m\angle EDA = \frac{\pi}{2} - \alpha$. This implies that H lies on the angle bisector of $\angle EDF$. By similar reasoning, H lies on the angle bisector of $\angle FED \implies H$ is the incenter of $\triangle DEF$. Q.E.D.

5. (USAMO 1990) $\triangle ABC$ is acute. The circle with diameter \overline{AB} intersects altitude $\overline{CC'}$ and its extensions at points M and N , and the circle with diameter \overline{AC} intersects altitude $\overline{BB'}$ and its extensions at P and Q . Prove that the points M, N, P , and Q lie on a common circle.

Solution

$\angle ANB$ and $\angle BMA$ both intercept diameter \overline{AB} , so both are right angles. We have $ANBM$, a cyclic quadrilateral, with two right angles opposite each other and perpendicular diagonals, so $ANBM$ is a kite in which $AM = AN$. Similarly, $AP = AQ$.

Because $\angle AC'C$ is right, $AC' = AC \cdot \cos A = b \cos A$. Let the center of the circumcircle of $ANBM$ be O_1 . The radius of this circle is $\frac{AB}{2} = \frac{c}{2}$. Because $\overline{O_1M}$ is a radius, $O_1M = \frac{c}{2}$. Because C' lies on radius O_1A , $O_1C' = \frac{c}{2} - b \cos A$. Let $CM = h$. By Pythagoras:

$$\begin{aligned} \left(\frac{c}{2}\right)^2 &= \left(\frac{c}{2} - b \cos A\right)^2 + h^2 \\ AM^2 &= (b \cos A)^2 + h^2 \\ \implies \left(\frac{c}{2}\right)^2 + (b \cos A)^2 &= \left(\frac{c}{2} - b \cos A\right)^2 + AM^2 \\ \iff AM &= \sqrt{2bc \cos A} \end{aligned}$$

Because of the symmetric nature of this problem, a similar argument applied to $APCQ$ yields $AP = \sqrt{2bc \cos A} = AM \implies AN = AM = AP = AQ \iff M, P, N,$ and Q are four points that lie on a circle centered at A . Q.E.D.