TJ USAMO Practice 6

VMT Math Team

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1. Let $\triangle ABC$ be equilateral, and let D and E be points on $\overline{AB}$ and $\overline{AC}$, respectively, such that $AD = CE$. Let $\overline{BE}$ and $\overline{CD}$ meet at F. If $[ABC] = 7$ and $[BCF] = 2$, compute $\frac{BD}{DA}$.

2. In $\triangle ABC$, $\angle B = 2\angle C$. Prove that $AC^2 = AB^2 + AB \cdot BC$.

3. Let I be the incenter of $\triangle ABC$, and A' the midpoint of $\breve{BC}$ of the circumcircle of $\triangle ABC$. Prove that $A'B = A'C = A'I$.

4. Let D, E, and F be the feet of the altitudes from A, B, and C, respectively, in $\triangle ABC$. If H is the orthocenter of $\triangle ABC$, then prove that $\triangle AFE$, $\triangle BDF$, and $\triangle CDE$ are all similar to $\triangle ABC$, and that H is the incenter of $\triangle DEF$.

5. (USAMO 1990) $\triangle ABC$ is acute. The circle with diameter $\overline{AB}$ intersects altitude $\overline{CC'}$ and its extensions at points M and N, and the circle with diameter $\overline{AC}$ intersects altitude $\overline{BB'}$ and its extensions at P and Q. Prove that the points M, N, P, and Q lie on a common circle.