TJ USAMO Practice 7 Solutions

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1. (IMO 1984) For $x, y, x > 0$ and $x + y + z = 1$, prove that $xy + yz + zx - 2xyz \leq \frac{7}{27}$.

   Solution

   We first prove a Lemma: If x and y are two real numbers with $x \leq y$, increasing x by some non-negative amount r and decreasing y by the same amount such that $r \leq \frac{y-x}{2}$ gives x' and y' with $x'y' \geq xy$. To verify this, we set $x = k - a$ and $y = k + a$. Then we have $x' = k - a + r$ and $y' = k + a - r$. The claim is then equivalent to $k^2 - a^2 \leq k^2 - (a - r)^2$, where $0 \leq r \leq a$. This is clearly true.

   We assume that $x \geq y \geq z \Longrightarrow z \leq \frac{1}{3}$. We have $xy + yz + zy - 2xyz = z(x + y) + xy(1-2z) \leq z(1-z)+(\frac{x+y}{2})^2(1-2z)$. This follows from the fact that $1-2z$ is positive and the lemma. Thus, we can increase the left hand side if the two largest numbers, x and y, are both set to their average. Because z is positive, $\frac{x+y}{2} < \frac{1}{2}$, so we can further increase the left hand side. Because until $x = y = z = \frac{1}{3}$, at least one of x, y, and z must be less than $\frac{1}{3}$, so we can increase that number to $\frac{1}{3}$ and decrease the larger numbers by the same total amount, all the while increasing the LHS. Thus, maximum must occur at $x = y = z = \frac{1}{3}$, which yields the equality case, and we are done.

2. Let $ABCDEF$ be a hexagon inscribed in the unit circle with center O such that the major diagonals of $ABCDEF$ pass through O. Find, with proof, the maximum value of $[AOB] + [COD] + [EOF]$ where $[AOB]$ denotes the area of $\triangle AOB$.

   Solution

   Let $\alpha = m\angle AOP = m\angle DOE$, $\beta = m\angle BOC = m\angle EOF$, and $\gamma = m\angle COD = m\angle FOA$. Because the sum of the six angles is 2π, we have $\alpha + \beta + \gamma = \pi$. Because O is the center of the unit circle, we have $OA = OB = OC = OD = OE = OF = 1$. By Jensen, it follows that the maximum area is $\frac{3\sqrt{3}}{4}$, obtained only when $ABCDEF$ is a regular hexagon:

$$[AOB] + [COD] + [EOF] \;=\; \frac{1}{2}(1^2 \sin\alpha + 1^2 \sin\beta + 1^2 \sin\gamma)$$

$$
\begin{aligned}
&= \frac{\sin \alpha + \sin \beta + \sin \gamma}{2} \\
&\leq \frac{1}{2} \cdot 3 \cdot \sin\left(\frac{\alpha+\beta+\gamma}{3}\right) = \frac{3\sqrt{3}}{4}
\end{aligned}
$$

3. Given that α, β, and γ are the angles of a triangle, show that

$$0 \leq \sin(\alpha+\beta-\gamma) + \sin(\beta+\gamma-\alpha) + \sin(\gamma+\alpha-\beta) \leq \frac{3\sqrt{3}}{2}$$

Solution

We have $\alpha + \beta = \pi - \gamma$. So we have

$$
\begin{aligned}
\sin(\alpha+\beta-\gamma) + \sin(\beta+\gamma-\alpha) + \sin(\gamma+\alpha-\beta) &= \sin(\pi-2\gamma) + \sin(\pi-2\alpha) + \sin(\pi-2\beta) \\
&= \sin(2\alpha) + \sin(2\beta) + \sin(2\gamma)
\end{aligned}
$$

Because $-1 \leq \sin(x) \leq 1$, if the sum exceeds $\frac{3\sqrt{3}}{2}$, then we must have $\sin(2\alpha), \sin(2\beta), \sin(2\gamma) > 0$, thus we may assume that $0 \leq 2\alpha, 2\beta, 2\gamma \leq \pi$. Because $\sin(x)$ is concave on the interval $(0, \pi)$, the upper bound follows from Jensen.

For the sum to be negative, at least one of the sines must be negative. Because of the restriction $\alpha + \beta + \gamma = \pi$, at most one of α, β, γ exceeds $\frac{\pi}{2}$. WLOG, this is γ. Then we must have $\alpha, \beta \in (0, \frac{\pi}{2}) \iff 2\alpha, 2\beta \in (0, \pi)$. We also have the restriction that $2\alpha + 2\beta \leq \pi$ because $\gamma \geq \frac{\pi}{2}$. Because sine is concave on the interval $(0, \pi)$, we have $\sin(2\alpha) + \sin(2\beta) \geq \sin(0) + \sin(2\alpha + 2\beta) = \sin(2\pi - 2\gamma) = -\sin(2\gamma)$. Adding $\sin(2\gamma)$ to these inequalities yields the desired result. Q.E.D.

4. In $\triangle ABC$, let $a = BC$ and $b = CA$, and let l_a and l_b denote the lengths of the internal angle bisectors of $\angle A$ and $\angle B$ respectively. Find the smallest k such that

$$\frac{l_a + l_b}{a+b} \leq k$$

Solution

Let D be the point on $\overline{AC}$: $\overline{BD}$ bisects $\angle ABC$, and let $c = AB$. We have $AD = \frac{bc}{a+c}$ and $DC = \frac{ba}{a+c}$, which follows from the angle-bisector theorem. Next, an application of Stewart's Theorem yields:

$$
\begin{aligned}
\left(\frac{bc}{a+c}\right)(b)\left(\frac{ba}{a+c}\right) + bl_b^2 &= \left(\frac{bc}{a+c}\right)a^2 + \left(\frac{ba}{a+c}\right)c^2 \\
\iff bl_b^2 &= \frac{abc}{a+c} \cdot (a+c) - \frac{ab^3c}{(a+c)^2} \\
\iff l_b^2 &= ac \cdot \left(1 - \frac{b^2}{(a+c)^2}\right)
\end{aligned}
$$

Similarly, $l_a^2 = bc \cdot (1 - \frac{a^2}{(b+c)^2})$. We see that increasing c increases l_a^2 and l_b^2, which, because l_a and l_b are positive, increases l_a and l_b. This in turn increases $\frac{l_a+l_b}{a+b}$. Because a, b, and c are restricted to be sides of a triangle, we increase c to its maximum and set $c = a + b$.

The substitution allows us to simplify the expressions for l_a and l_b considerably:

$$\begin{aligned} l_a &= \sqrt{ac \cdot (1 - \frac{b^2}{(a+c)^2})} \\ &= \sqrt{a(a+b) \cdot (\frac{(2a+b)^2 - b^2}{(2a+b)^2})} \\ &= \sqrt{a(a+b) \cdot \frac{(4 \cdot a \cdot (a+b))}{(2a+b)^2}} \\ &= \frac{2a(a+b)}{2a+b} \end{aligned}$$

Subsituting this into the desired expression, we obtain:

$$\begin{aligned} \frac{l_a + l_b}{a+b} &= \frac{2a}{2a+b} + \frac{2b}{2b+a} \\ &= \frac{2a(2b+a) + 2b(2a+b)}{(2a+b)(2b+a)} \\ &= \frac{2a^2 + 8ab + 2b^2}{2a^2 + 5ab + 2b^2} \\ &= 1 + \frac{3ab}{2a^2 + 5ab + 2b^2} \\ &\leq 1 + \frac{3ab}{9ab} = \frac{4}{3} \end{aligned}$$

Thus, we can increase the expression $\frac{l_a+l_b}{a+b}$ to a maximum of $\frac{4}{3} \Longrightarrow$ the smallest k is $\frac{4}{3}$.

5. (USAMO 1998) Let $a_0, a_1, \ldots, a_n$ be numbers in the interval $(0, \frac{\pi}{2})$ such that

$$\tan(a_0 - \frac{\pi}{4}) + \tan(a_1 - \frac{\pi}{4}) + \cdots + \tan(a_n - \frac{\pi}{4}) \geq n - 1$$

Prove that:

$$\tan(a_0)\tan(a_1)\cdots\tan(a_n) \geq n^{n+1}$$

Solution

We set $t_i = \tan(a_i - \frac{\pi}{4})$. Then we have $\tan(a_i) = \tan(\frac{\pi}{4} + (a_i - \frac{\pi}{4})) = \frac{1+t_i}{1-t_i}$. Thus, we wish to show that

$$\prod_{i=0}^{n}(\frac{1+t_i}{1-t_i}) \geq n^{n+1}$$

We have:

$$\begin{aligned}
t_1 + t_2 + \cdots + t_{n+1} &\geq n - 1 \\
\iff 1 + t_i &\geq \sum_{j \neq i}(1 - t_j) \\
\implies \frac{1+t_i}{n} &\geq \prod_{j \neq i}(1 - t_j)^{\frac{1}{n}} \\
\implies \frac{\prod_{i=0}^{n}(1+t_i)}{n^{n+1}} \geq \prod_{i=0}^{n}\prod_{j \neq i}(1 - t_j)^{\frac{1}{n}} &= \prod_{i=0}^{n}(1 - t_i)
\end{aligned}$$

From which the desired result follows. Q.E.D.