TJ USAMO Practice 7

VMT Math Team

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1. (IMO 1984) For $x, y, x > 0$ and $x + y + z = 1$, prove that $xy + yz + zx - 2xyz \leq \frac{7}{27}$.

2. Let $ABCDEF$ be a hexagon inscribed in the unit circle with center O such that the major diagonals of $ABCDEF$ pass through O. Find, with proof, the maximum value of $[AOB] + [COD] + [EOF]$ where $[AOB]$ denotes the area of $\triangle AOB$.

3. Given that α, β, and γ are the angles of a triangle, show that

$$0 \leq \sin(\alpha + \beta - \gamma) + \sin(\beta + \gamma - \alpha) + \sin(\gamma + \alpha - \beta) \leq \frac{3\sqrt{3}}{2}$$

4. In $\triangle ABC$, let $a = BC$ and $b = CA$, and let l_a and l_b denote the lengths of the internal angle bisectors of $\angle A$ and $\angle B$ respectively. Find the smallest k such that

$$\frac{l_a + l_b}{a + b} \leq k$$

5. (USAMO 1998) Let $a_0, a_1, \ldots, a_n$ be numbers in the interval $(0, \frac{\pi}{2})$ such that

$$\tan(a_0 - \frac{\pi}{4}) + \tan(a_1 - \frac{\pi}{4}) + \cdots + \tan(a_n - \frac{\pi}{4}) \geq n - 1$$

   Prove that:

$$\tan(a_0)\tan(a_1)\cdots\tan(a_n) \geq n^{n+1}$$