TJ USAMO Practice 8

VMT Math Team

December 22, 2003

1. Let $p_1, p_2, p_3, \ldots$ be the prime numbers listed in increasing order, and let x_0 be a real number between 0 and 1. For positive integer k, define $x_k = 0$ if $x_{k-1} = 0$, $x_k = \left\{\frac{p_k}{x_{k-1}}\right\}$ otherwise, where $\{x\}$ denotes the fractional part of x. (The fractional part of x is given by $x - \lfloor x \rfloor$ where $\lfloor x \rfloor$ is the greatest integer less than or equal to x.) Find, with proof, all x_0 satisfying $0 < x_0 < 1$ for which the sequence $x_0, x_1, x_2, \ldots$ eventually becomes 0.

   Solution

   The answer is all rational numbers. Note that if x_k is irrational then $x_{k+1} = \frac{p_{k+1}}{x_k} - \lfloor \frac{p_{k+1}}{x_k} \rfloor$ = (irrational) - (rational) is irrational. If x_0 is irrational then x_k is irrational for all k, so since 0 is rational x_k can't be 0. Therefore x_0 must be rational if the sequence becomes 0 at some point.

   If x_k is rational and non-zero, then let $x_k = \frac{q}{r}$, where q and r are positive integers and $q < r$. Then $x_{k+1} = \left\{\frac{p_{k+1}r}{q}\right\} = \frac{s}{q}$, where $s < q < r$. Therefore the numerator strictly decreases over the sequence, so at some point it must reach 0, QED.

2. Let ABC be a triangle, and draw isosceles triangles BCD, CAE, ABF externally to ABC, with BC, CA, AB as their respective bases. Prove that the lines through A, B, C perpendicular to the lines $\overleftrightarrow{EF}, \overleftrightarrow{FD}, \overleftrightarrow{DE}$, respectively, are concurrent.

   Solution

   Let H, I, J be the intersections of the perpindiculars from A, B, C to $\overleftrightarrow{EF}, \overleftrightarrow{FD}, \overleftrightarrow{DE}$, respectively, and K, L, M be the midpoints of BC, AC, AB, respectively. Then $\angle AMF = \angle AHF = \pi/2$, so $AHMF$ is cyclic, so $\angle HAM = \angle HFM$. Similarly, $\angle MFI = \angle MBI, \angle IBK = \angle IDK, \angle KDJ = \angle KCJ, \angle JCL = \angle JEL, \angle LEH = \angle LAH$. Because DK, EL, FM are the perpindicular bisectors of the sides, they concur at the circumcenter of ABC. Therefore, by trig Ceva,

$$\frac{\sin HFM}{\sin MFI} \cdot \frac{\sin IDK}{\sin KDJ} \cdot \frac{\sin JEL}{\sin LEH}.$$

or by substitution

$$\frac{\sin HAM}{\sin MBI} \cdot \frac{\sin IBK}{\sin KCJ} \cdot \frac{\sin JCL}{\sin LAH}.$$

So by trig ceva, CJ, AH, BI concur, QED.

3. Prove that, for all positive real numbers x, y, z such that $x^2 + y^2 + z^2 + xyz = 4$,

$$xyz \leq xy + yz + zx \leq xyz + 2$$

Solution

Without loss of generality, $0 \leq x \leq y \leq z \leq 2$. By the constraint, $x \leq 1 \leq z$, because otherwise $x^2 + y^2 + z^2 + xyz < 1+1+1+1 = 4$ or $x^2 + y^2 + xyz > 1+1+1+1 = 4$. For the left hand side, $xy + yz + zx \geq yz \geq xyz$, what we wanted.

Note that $z \leq 2 - xy$, because at equality, $x^2 + y^2 + z^2 + xyz = x^2 + y^2 + 4 - 4xy + x^2y^2 + 2xy - x^2y^2 = 4 + (x-y)^2 \geq 4$, so an increase in z would make the condition false. Similarly, $x \leq 2 - yz$. $xy + yz + zx \leq xyz + 2$ iff $z(x + y - xy - 1) \leq 2 - xy - z$, or $z(x-1)(1-y) \leq 2 - xy - z$. If $y \leq 1$ then this is true, because $z(x-1)(1-y) \leq 0$ because one factor is negative and $0 \leq 2 - xy - z$ from above, so we're done. If $y \geq 1$, then similarly $xy + yz + zx \leq xyz + 2$ iff $x(z-1)(1-y) \leq 2 - yz - x$, which is again true because the left is nonpositive and the right nonnegative, QED.