TJ USAMO Practice 8

VMT Math Team

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1. Let $p_1, p_2, p_3, \ldots$ be the prime numbers listed in increasing order, and let x_0 be a real number between 0 and 1. For positive integer k, define $x_k = 0$ if $x_{k-1} = 0$, $x_k = \left\{ \frac{p_k}{x_{k-1}} \right\}$ otherwise, where $\{x\}$ denotes the fractional part of x. (The fractional part of x is given by $x - \lfloor x \rfloor$ where $\lfloor x \rfloor$ is the greatest integer less than or equal to x.) Find, with proof, all x_0 satisfying $0 < x_0 < 1$ for which the sequence $x_0, x_1, x_2, \ldots$ eventually becomes 0.

2. Let ABC be a triangle, and draw isosceles triangles BCD, CAE, ABF externally to ABC, with BC, CA, AB as their respective bases. Prove that the lines through A, B, C perpendicular to the lines $\overleftrightarrow{EF}, \overleftrightarrow{FD}, \overleftrightarrow{DE}$, respectively, are concurrent.

3. Prove that, for all positive real numbers x, y, z such that $x^2 + y^2 + z^2 + xyz = 4$,

$$xyz \leq xy + yz + zx \leq xyz + 2$$