TJ USAMO Practice 9 - Geometry II

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Warm-Up: Let A, B, C, and D be four points that lie on a line in that order, with $AB = CD$. Show that for any point P, $AP + DP \geq BP + CP$.

Solution

Let M be the midpoint of $\overline{BC}$, and rotate P 180 degrees about M to P'. By symmetry, we have $AP = DP'$, $BP = CP'$, $CP = BP'$ and $DP = AP'$. It follows that $AP + DP \geq BP + CP \iff AP + DP + AP' + DP' \geq BP + CP + BP' + CP \iff AP + AP' \geq BP + BP'$. To show this, we extend $\overline{PB}$ past B to B' on $\overline{AP'}$. Now, by the triangle inequality, we have $AP + AB' > B'P$ and $P'B' + B'B > BP'$. It follows that

$$\begin{aligned} AP + AP' &= AP + AB' + B'P' > B'P + B'P' \\ &= B'B + BP + B'P' > BP + BP' \end{aligned}$$

From which the desired result follows.

1. $\triangle ABC$ is an equilateral triangle of side length s. P is a point on the circumcircle of $\triangle ABC$, such that P is on the minor arc $\breve{AC}$. Let X be the intersection of $\overline{AC}$ and $\overline{BP}$. Show that

$$BP = AP + CP \qquad (1)$$

$$XP = \frac{(AP + CP) \cdot AP \cdot CP}{AP \cdot CP + s^2} \qquad (2)$$

Solution

The fact that $\triangle ABC$ is equilateral implies that $AB = BC = CA$, and substitution of this into Ptolemy's Theorem applied to $ABCD$ yields $AB * AP + AB * CP = AB * BP$, and dividing out AB yields (1).

We now argue that $\frac{XP}{XB} = \frac{[APC]}{[ABC]}$. This is true because $[APC] = \frac{1}{2} XP \cdot AC \cdot \sin \angle AXP$, $[ABC] = \frac{1}{2} XB \cdot AC \cdot \sin \angle CXB$, and $\angle AXP \cong \angle CXB$. Other ways of computing $[ABC]$ and $[APC]$ are $[ABC] = \frac{1}{2} AB \cdot BC \cdot \sin \angle ABC$ and $[APC] = \frac{1}{2} AP \cdot PC \cdot \sin \angle CPA$. Because $ABCD$ is a cyclic quadrilateral, $m\angle ABC + m\angle CPA = \pi \iff$

$\sin \angle ABC = \sin \angle CPA$. Thus, $\frac{XP}{XB} = \frac{[APC]}{[ABC]} = \frac{AP \cdot PC}{AB \cdot BC}$.

Because $XP + XB = BP$, it follows that $XP(1 + \frac{AB \cdot BC}{AP \cdot PC}) = BP = AP + CP$. Substituting $s = AB = BC$ and solving for XP yields $XP = \frac{(AP+CP) \cdot AP \cdot CP}{AP \cdot CP + s^2}$. Q.E.D.

2. A, B, and C are points on circle ω. M is the midpoint of $\overline{AB}$, and D is the other intersection of $\overline{CM}$ and ω. Let the tangents to ω from C and D meet the extensions of $\overline{AB}$ at P and Q respectively. Show that $CP = DQ$.

   Solution

   Let O denote the center of ω.[1] $\overline{OM} \perp \overline{AB}$ because M is the mipoint of chord $\overline{AB}$ and $overline OC \perp \overline{PC}$ because $\overline{PC}$ is tangent to ω. Because $\overline{DQ}$ is also tangent to ω, $\overline{OD} \perp \overline{DQ}$.

$$\begin{aligned} \angle OCP \cong \angle OMP &\implies OMCP \text{ cyclic} \implies \angle MPO \cong \angle MCO \\ \angle QMO \cong \angle ODQ &\implies ODQM \text{ cyclic} \implies \angle OQM \cong \angle ODM \\ \overline{OC} \text{ and } \overline{OD} \text{ are radii} &\implies OC = OD \implies \angle ODC \cong \angle DCO \\ &\implies \angle QPO \cong \angle OQP \end{aligned}$$

   It follows that $OP = OQ$. Because $\triangle OCP$ and $\triangle ODQ$ are right, $OC^2 + PC^2 = OP^2 = OQ^2 = OD^2 + DQ^2 \implies PC^2 = DQ^2 \implies PC = DQ$. Q.E.D.

3. (Iran) ω_1 and ω_2 are circles with centers O_1 and O_2 (respectively) that meet at points A and B. The radii $\overline{O_1B}$ and $\overline{O_2B}$ meet circles ω_2 and ω_1 at C and D. The line through B that is parallel to $\overline{CD}$ intersects ω_1 and ω_2 at M and N. Show that $AC + AD = MN$.

   Solution

   $O_1D = O_1B \implies \angle O_1DB \cong \angle DBO_1$. Similarly, $\angle O_2BC \cong \angle BCO_2 \implies \angle O_1DB \cong \angle BCO_2 \implies \angle O_2CO_1 \cong \angle O_2DO_1 \implies O_1O_2DC$ cyclic.

   Because ADBM is cyclic, $m\angle ADB = \pi - m\angle BMA = \pi - \frac{1}{2} m\angle BO_1A$. By symmetry, $m\angle BO_1O_2 = m\angle O_2O_1A = \frac{1}{2} m\angle BO_1A \implies \pi - \frac{1}{2} m\angle BO_1A = \pi - m\angle BO_1O_2 = \pi - m\angle CO_1O_2$. We know that O_1O_2DC is cyclic, so $\pi - m\angle CO_1O_2 = m\angle CDO_2$. Finally, because $\overline{CD} \parallel \overline{MN}$, $m\angle CDO_2 = m\angle DBM$.

[1] It is almost always beneficial to consider the center of a circle because it is involved in many right angles, which in turn lead to cyclic quadrilaterals and other useful facts.

Combining these, we have $\angle ADB \cong \angle DBM \implies ADBM$ is an isosceles trapezoid $\implies MB = AD$. Similarly, $BN = AC$. Adding produces the desired result.[2]

4. (Hong Kong) P is a point in the interior of $\triangle ABC$ such that $AP = AC$ and $BP = CP$. If $m\angle C = 2m\angle B$, show that $\overline{AP}$ trisects $\angle A$.

Solution

Let D be the midpoint of $\overline{BC}$ and E be the point on $\overline{AB} : \overline{CE}$ bisects $\angle C$. Because $m\angle C = 2m\angle B$, $\frac{1}{2}m\angle C = m\angle B \implies \angle CBE \cong \angle ECB$. It follows that D, P, and E are collinear in that order.

By the Exterior Angle Theorem, $m\angle CEA = m\angle ECB + m\angle CBE \implies \triangle ACE \sim \triangle ABC$. Armed with the given that $AP = AC$, this similarity gives $\frac{AE}{AC} = \frac{AC}{AB} \implies \frac{AE}{AP} = \frac{AP}{AB} \implies \triangle APE \sim \triangle ABP \implies \angle APE \cong \angle ABP$.
Let $m\angle PBE = x$ and $m\angle DBP = y$. Then we have $m\angle PCD = y \implies m\angle DPC = \frac{\pi}{2} - y, m\angle ECP = x \implies m\angle ACE = x + y$. Because $AP = AC$, $\triangle ACP$ is isosceles, $\implies m\angle CPA = 2x + y \implies m\angle PAC = \pi - 4x - 2y \implies m\angle BAP = x - y$. Because D, P, and E are collinear, $m\angle APE = \frac{\pi}{2} - x$.

Earlier we showed that $\angle APE \cong \angle ABP$, so we must have $x = \frac{\pi}{2} - 2x \implies x = \frac{\pi}{6} \implies m\angle BAP = \frac{\pi}{6} - y$, $m\angle PAC = \pi - 4 \cdot \frac{\pi}{6} - 2y = \frac{\pi}{3} - 2y = 2m\angle BAP$. Q.E.D.

[2] After a *well-drawn* few diagrams, it becomes apparent that $ADBM$ and $ACBN$ are isosceles trapezoids. Knowing this enables us to have a direction in mind as we work out the details.