TJ USAMO Practice 9 - Geometry II

VMT Math Team

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Warm-Up: Let A, B, C, and D be four points that lie on a line in that order, with $AB = CD$. Show that for any point P, $AP + DP \geq BP + CP$.

1. $\triangle ABC$ is an equilateral triangle of side length s. P is a point on the circumcircle of $\triangle ABC$, such that P is on the minor arc $\breve{AC}$. Let X be the intersection of $\overline{AC}$ and $\overline{BP}$. Show that

$$BP = AP + CP \qquad (1)$$

$$XP = \frac{(AP + CP) \cdot AP \cdot CP}{AP \cdot CP + s^2} \qquad (2)$$

2. A, B, and C are points on circle ω. M is the midpoint of $\overline{AB}$, and D is the other intersection of $\overline{CM}$ and ω. Let the tangents to ω from C and D meet the extensions of $\overline{AB}$ at P and Q respectively. Show that $CP = DQ$.

3. (Iran) ω_1 and ω_2 are circles with centers O_1 and O_2 (respectively) that meet at points A and B. The radii $\overline{O_1B}$ and $\overline{O_2B}$ meet circles ω_2 and ω_1 at C and D. The line through B that is parallel to $\overline{CD}$ intersects ω_1 and ω_2 at M and N. Show that $AC + AD = MN$.

4. (Hong Kong) P is a point in the interior of $\triangle ABC$ such that $AP = AC$ and $BP = CP$. If $m\angle C = 2m\angle B$, show that $\overline{AP}$ trisects $\angle A$.