The Art of Dumbassing

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1 Introduction

The field of inequalities is far from trivial. Many techniques, including Cauchy, Hölder, isolated fudging, Jensen, and smoothing exist and lead to simple, short solutions. That is not the purpose of the technique that I will be describing below. This technique is very straightforward, widely (but not universally) applicable, and can lead to a solution quickly, although it is almost never the nicest one. This technique is known as dumbassing, and is quite aptly named, as very little creativity is required. The crux of this method is homogenizing, clearing denominators, multiplying everything out, then finding the correct applications of Schur and/or AM-GM. Occaisionally, more advanced techniques may be required.

2 Chinese Dumbass Notation

The first and foremost purpose of this lecture is to introduce a relatively obscure but extremely useful notation. Before I describe it, I want you to be sure to realize that this notation is *not* standard. If you are going to use it on a contest, be sure to define it. With that warning in mind, allow me to describe the notation.

Chinese Dumbass Notation is a concise and convenient way to write down three variable homogeneous expressions. Additionally, it provides a convenient way to multiply out two expressions with very little chance of dropping terms. Unlike cyclic and symmetric sum notation, Chinese Dumbass Notation allows one to keep like terms combined while not requiring any sort of symmetry from the expression. However, let's first look at an example that is symmetric.

$$\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & 8 & & 8 & & & \\
 & & 20 & & 28 & & 20 & & \\
 & 8 & & 28 & & 28 & & 8 & \\
0 & & 8 & & 20 & & 8 & & 0
\end{array}$$

The first thing that one should note about this notation is that it is not linear. Chinese Dumbass Notation uses a triangle to hold the coefficients of a three variable symmetric inequality in a way that is easy to visualize. The above triangle represents the expression $\sum_{sym} 8a^3b + \sum_{sym} 10a^2b^2 + \sum_{sym} 14a^2bc$. In general, a fourth degree expression would be written as follows.

$$\begin{array}{ccccccccc}
 & & & & [a^4] & & & & \\
 & & & [a^3b] & & [a^3c] & & & \\
 & & [a^2b^2] & & [a^2bc] & & [a^2c^2] & & \\
 & [ab^3] & & [ab^2c] & & [abc^2] & & [ac^3] & \\
[b^4] & & [b^3c] & & [b^2c^2] & & [bc^3] & & [c^4]
\end{array}$$

Here $[x]$ represents the coefficient of x. In general, a degree d expression is represented by a triangle with side length $d+1$ and the coefficient of $a^\alpha b^\beta c^\gamma$ is placed at Barycentric coordinates (α, β, γ). Multiplying these triangles is much like multiplying polynomials: you do so by shifting and adding. For example,

$$\begin{pmatrix} & 1 & \\ 1 & & 2 \end{pmatrix}\begin{pmatrix} & 1 & \\ 2 & & 1 \end{pmatrix} = 1\begin{pmatrix} & & 1 & & \\ & 1 & & 2 & \\ 0 & & 0 & & 0 \end{pmatrix} + 2\begin{pmatrix} & & 0 & & \\ & 1 & & 0 & \\ 1 & & 2 & & 0 \end{pmatrix} + 1\begin{pmatrix} & & 0 & & \\ & 0 & & 1 & \\ 0 & & 1 & & 2 \end{pmatrix}$$

$$= \begin{array}{ccccc} & & 1 & & \\ & 3 & & 3 & \\ 2 & & 5 & & 2 \end{array}$$

It is possible to shortcut this process much like how you shortcut polynomial multiplications. The main idea is that when you want to compute a coefficient in the answer triangle, you look at all the ways it can be made in the factor triangles. Practicing this method will allow you to learn what form of multiplication works best for you. Now that we have the notation down, let's look at some basic inequalities.

3 AM-GM

The AM-GM inequality for two variables states that $a^2+b^2 \geq 2ab$. In other words, $\begin{array}{ccccc} & & 1 & & \\ & -2 & & 0 & \\ 1 & & 0 & & 0 \end{array} \geq 0$. Of course, this works anywhere in the triangle, even for spaces that aren't consecutive. In general, weighted AM-GM allows us to take some positive coefficients and, thinking of them as weights, "slide" them to their center of mass while not increasing the total sum of the expression. One extremely important application of this is to take a symmetric distribution of weights and slide them inward to another symmetric distribution of weights. The fact that these inequalities follow from AM-GM is known as Muirhead's Inequality. For example,

$$\begin{array}{ccccccccc} & & & & 0 & & & & \\ & & & 1 & & 1 & & & \\ & & 0 & & 0 & & 0 & & \\ & 1 & & 0 & & 0 & & 1 & \\ 0 & & 1 & & 0 & & 1 & & 0 \end{array} \geq \begin{array}{ccccccccc} & & & & 0 & & & & \\ & & & 0 & & 0 & & & \\ & & 0 & & 2 & & 0 & & \\ & 0 & & 2 & & 2 & & 0 & \\ 0 & & 0 & & 0 & & 0 & & 0 \end{array}$$

follows easily from Muirhead's Inequality. We can see that in this case it also follows from a symmetric sum of the following application of weighted AM-GM:

$$\begin{array}{ccccccccc} & & & & 0 & & & & \\ & & & \frac{2}{3} & & 0 & & & \\ & & 0 & & 0 & & 0 & & \\ & 0 & & 0 & & 0 & & 0 & \\ 0 & & 0 & & 0 & & \frac{1}{3} & & 0 \end{array} \geq \begin{array}{ccccccccc} & & & & 0 & & & & \\ & & & 0 & & 0 & & & \\ & & 0 & & 1 & & 0 & & \\ & 0 & & 0 & & 0 & & 0 & \\ 0 & & 0 & & 0 & & 0 & & 0 \end{array}$$

In general, citing Muirhead's inequality is looked down upon, but it simplifies many dumbassing arguments considerably as they are often a sum of many applications of AM-GM and finding the weights for each one takes considerable time.

4 Schur's Inequality

Schur's inequality states that $\sum_{cyc} a^r(a-b)(a-c) \geq 0$ for all nonnegative numbers a, b, c, r. The prototypical application of Schur's inequality looks as follows:

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & -1 & & -1 & & & \\ & & 0 & & 1 & & 0 & & \\ & -1 & & 1 & & 1 & & -1 & \\ 1 & & -1 & & 0 & & -1 & & 1 \end{array} \geq 0$$

As with AM-GM, the variables can be tweaked so that Schur's inequality gives us $\sum_{cyc} a^r(a^s - b^s)(a^s - c^s) \geq 0$. In Chinese Dumbass Notation, this corresponds to manipulating the characteristic diamonds shown above, with r controlling the distance from the center and s controlling the size of each of the diamonds. The inequality shown above is with $r = 2$ and $s = 1$.

5 Examples

Example 1. Prove that $\dfrac{bc}{2a+b+c} + \dfrac{ac}{a+2b+c} + \dfrac{ab}{a+b+2c} \leq \dfrac{1}{4}(a+b+c)$.

Proof. We clear denominators to obtain that this is equivalent to

$$4\sum_{cyc} bc(a+2b+c)(a+b+2c) \leq (a+2b+c)(a+b+2c)(2a+b+c)(a+b+c)$$

$$\Leftrightarrow 4\sum_{cyc}\left(\begin{array}{ccccc} & & 0 & & \\ & 0 & & 0 & \\ 0 & & 1 & & 0 \end{array}\right)\left(\begin{array}{ccc} & 1 & \\ 1 & & 2 \end{array}\right)\left(\begin{array}{ccc} & 1 & \\ 2 & & 1 \end{array}\right) \leq \left(\begin{array}{ccc} & 1 & \\ 1 & & 2 \end{array}\right)\left(\begin{array}{ccc} & 1 & \\ 2 & & 1 \end{array}\right)\left(\begin{array}{ccc} & 2 & \\ 1 & & 1 \end{array}\right)\left(\begin{array}{ccc} & 1 & \\ 1 & & 1 \end{array}\right)$$

We'll expand the left hand side as

$$4\sum_{cyc}\left(\begin{array}{ccccc} & & 0 & & \\ & 0 & & 0 & \\ 0 & & 1 & & 0 \end{array}\right)\left(\begin{array}{ccc} & 1 & \\ 1 & & 2 \end{array}\right)\left(\begin{array}{ccc} & 1 & \\ 2 & & 1 \end{array}\right)$$

$$= 4\sum_{cyc}\left(\begin{array}{ccccc} & & 0 & & \\ & 0 & & 0 & \\ 0 & & 1 & & 0 \end{array}\right)\left(\begin{array}{ccccc} & & 1 & & \\ & 3 & & 3 & \\ 2 & & 5 & & 2 \end{array}\right)$$

$$= 4\sum_{cyc}\left(\begin{array}{ccccccccc} & & & & 0 & & & & \\ & & & 0 & & 0 & & & \\ & & 0 & & 1 & & 0 & & \\ & 0 & & 3 & & 3 & & 0 & \\ 0 & & 2 & & 5 & & 2 & & 0 \end{array}\right)$$

$$= 4\left(\begin{array}{ccccccccc} & & & & 0 & & & & \\ & & & 2 & & 2 & & & \\ & & 5 & & 7 & & 5 & & \\ & 2 & & 7 & & 7 & & 2 & \\ 0 & & 2 & & 5 & & 2 & & 0 \end{array}\right)$$

$$= \begin{array}{ccccccccc} & & & & 0 & & & & \\ & & & 8 & & 8 & & & \\ & & 20 & & 28 & & 20 & & \\ & 8 & & 28 & & 28 & & 8 & \\ 0 & & 8 & & 20 & & 8 & & 0 \end{array}$$

We then expand the right hand side as

$$
\begin{aligned}
&\begin{pmatrix} & 1 & \\ 1 & & 2 \end{pmatrix}\begin{pmatrix} & 1 & \\ 2 & & 1 \end{pmatrix}\begin{pmatrix} & 2 & \\ 1 & & 1 \end{pmatrix}\begin{pmatrix} & 1 & \\ 1 & & 1 \end{pmatrix} \\
&= \begin{pmatrix} & & 1 & & \\ & 3 & & 3 & \\ 2 & & 5 & & 2 \end{pmatrix}\begin{pmatrix} & 2 & \\ 1 & & 1 \end{pmatrix}\begin{pmatrix} & 1 & \\ 1 & & 1 \end{pmatrix} \\
&= \begin{pmatrix} & & & 2 & & & \\ & & 7 & & 7 & & \\ & 7 & & 16 & & 7 & \\ 2 & & 7 & & 7 & & 2 \end{pmatrix}\begin{pmatrix} & 1 & \\ 1 & & 1 \end{pmatrix} \\
&= \begin{array}{ccccccccc} & & & & 2 & & & & \\ & & & 9 & & 9 & & & \\ & & 14 & & 30 & & 14 & & \\ & 9 & & 30 & & 30 & & 9 & \\ 2 & & 9 & & 14 & & 9 & & 2 \end{array}
\end{aligned}
$$

The inequality is thus equivalent to showing that the difference is nonnegative. But the difference is

$$
\begin{array}{ccccccccc} & & & & 2 & & & & \\ & & & 1 & & 1 & & & \\ & & -6 & & 2 & & -6 & & \\ & 1 & & 2 & & 2 & & 1 & \\ 2 & & 1 & & -6 & & 1 & & 2 \end{array}
\geq
\begin{array}{ccccccccc} & & & & 0 & & & & \\ & & & 3 & & 3 & & & \\ & & -6 & & 0 & & -6 & & \\ & 3 & & 0 & & 0 & & 3 & \\ 0 & & 3 & & -6 & & 3 & & 0 \end{array}
\geq 0
$$

by Schur and AM-GM. ☐

Example 2. Let a, b, c be positive real numbers such that

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = a + b + c.$$

Show that

$$\frac{1}{(2a+b+c)^2} + \frac{1}{(2b+c+a)^2} + \frac{1}{(2c+a+b)^2} \leq \frac{3}{16}.$$

Proof. Notice that the condition is equivalent to

$$ab + ac + bc = a^2bc + ab^2c + abc^2.$$

Additionally, by multiplying our desired inequality through by $16(2a+b+c)^2(2b+a+c)^2(2c+a+b)^2$, we see that it is equivalent to

$$16\sum_{cyc}(2b+c+a)^2(2c+a+b)^2 \leq 3(2b+c+a)^2(2c+a+b)^2(2a+b+c)^2$$

which is equivalent to

$$16\sum_{cyc}(2b+c+a)^2(2c+a+b)^2(a^2bc+ab^2c+abc^2) \leq 3(2b+c+a)^2(2c+a+b)^2(2a+b+c)^2(ab+bc+ac).$$

We will prove this inequality for all positive reals a, b, c. First, we rewrite the left hand side as

$$
16\left(\sum_{cyc}\left(\begin{array}{ccc} & 1 & \\ 2 & & 1\end{array}\right)^2\left(\begin{array}{ccc} & 1 & \\ 1 & & 2\end{array}\right)^2\right)\left(\begin{array}{ccccccccc} &&&&0&&&& \\ &&&0&&0&&& \\ &&0&&1&&0&& \\ &0&&1&&1&&0& \\ 0&&0&&0&&0&&0\end{array}\right)
$$

$$
=16\left(\sum_{cyc}\left(\begin{array}{ccccc} &&1&& \\ &4&&2& \\ 4&&4&&1\end{array}\right)\left(\begin{array}{ccccc} &&1&& \\ &2&&4& \\ 1&&4&&4\end{array}\right)\right)\left(\begin{array}{ccccccccc} &&&&0&&&& \\ &&&0&&0&&& \\ &&0&&1&&0&& \\ &0&&1&&1&&0& \\ 0&&0&&0&&0&&0\end{array}\right)
$$

$$
=16\left(\sum_{cyc}\begin{array}{ccccccccc} &&&&1&&&& \\ &&&6&&6&&& \\ &&13&&28&&13&& \\ &12&&42&&42&&12& \\ 4&&20&&33&&20&&4\end{array}\right)\left(\begin{array}{ccccccccc} &&&&0&&&& \\ &&&0&&0&&& \\ &&0&&1&&0&& \\ &0&&1&&1&&0& \\ 0&&0&&0&&0&&0\end{array}\right)
$$

$$
=16\left(\begin{array}{ccccccccc} &&&&9&&&& \\ &&&38&&38&&& \\ &&59&&112&&59&& \\ &38&&112&&112&&38& \\ 9&&38&&59&&38&&9\end{array}\right)\left(\begin{array}{ccccccccc} &&&&0&&&& \\ &&&0&&0&&& \\ &&0&&1&&0&& \\ &0&&1&&1&&0& \\ 0&&0&&0&&0&&0\end{array}\right)
$$

$$
=16\left(\begin{array}{ccccccccccccccccc} &&&&&&&&0&&&&&&&& \\ &&&&&&&0&&0&&&&&&& \\ &&&&&&0&&9&&0&&&&&& \\ &&&&&0&&47&&47&&0&&&&& \\ &&&&0&&97&&188&&97&&0&&&& \\ &&&0&&97&&283&&283&&97&&0&&& \\ &&0&&47&&188&&283&&188&&47&&0&& \\ &0&&9&&47&&97&&97&&47&&9&&0& \\ 0&&0&&0&&0&&0&&0&&0&&0&&0\end{array}\right)
$$

$$
=\begin{array}{ccccccccccccccccc} &&&&&&&&0&&&&&&&& \\ &&&&&&&0&&0&&&&&&& \\ &&&&&&0&&144&&0&&&&&& \\ &&&&&0&&752&&752&&0&&&&& \\ &&&&0&&1552&&3008&&1552&&0&&&& \\ &&&0&&1552&&4528&&4528&&1552&&0&&& \\ &&0&&752&&3008&&4528&&3008&&752&&0&& \\ &0&&144&&752&&1552&&1552&&752&&144&&0& \\ 0&&0&&0&&0&&0&&0&&0&&0&&0\end{array}
$$

Now we will rewrite the right hand side as

$$3\left(\begin{array}{*{3}{c}} & 1 & \\ 2 & & 1\end{array}\right)^2\left(\begin{array}{*{3}{c}} & 1 & \\ 1 & & 2\end{array}\right)^2\left(\begin{array}{*{3}{c}} & 2 & \\ 1 & & 1\end{array}\right)^2\left(\begin{array}{*{5}{c}} &&0&& \\ &1&&1& \\ 0&&1&&0\end{array}\right)$$

$$= 3\left(\begin{array}{*{5}{c}} &&1&& \\ &4&&2& \\ 4&&4&&1\end{array}\right)\left(\begin{array}{*{5}{c}} &&1&& \\ &2&&4& \\ 1&&4&&4\end{array}\right)\left(\begin{array}{*{5}{c}} &&4&& \\ &4&&4& \\ 1&&2&&1\end{array}\right)\left(\begin{array}{*{5}{c}} &&0&& \\ &1&&1& \\ 0&&1&&0\end{array}\right)$$

$$= 3\left(\begin{array}{*{9}{c}} &&&&1 \\ &&&6&&6 \\ &&13&&28&&13 \\ &12&&42&&42&&12 \\ 4&&20&&33&&20&&4\end{array}\right)\left(\begin{array}{*{5}{c}} &&4&& \\ &4&&4& \\ 1&&2&&1\end{array}\right)\left(\begin{array}{*{5}{c}} &&0&& \\ &1&&1& \\ 0&&1&&0\end{array}\right)$$

$$= 3\left(\begin{array}{*{13}{c}} &&&&&&4 \\ &&&&&28&&28 \\ &&&&77&&162&&77 \\ &&&106&&350&&350&&106 \\ &&77&&350&&550&&350&&77 \\ &28&&162&&350&&350&&162&&28 \\ 4&&28&&77&&106&&77&&28&&4\end{array}\right)\left(\begin{array}{*{5}{c}} &&0&& \\ &1&&1& \\ 0&&1&&0\end{array}\right)$$

$$= 3\left(\begin{array}{*{17}{c}} &&&&&&&&0 \\ &&&&&&&4&&4 \\ &&&&&&28&&60&&28 \\ &&&&&77&&267&&267&&77 \\ &&&&106&&533&&862&&533&&106 \\ &&&77&&533&&1250&&1250&&533&&77 \\ &&28&&267&&862&&1250&&862&&267&&28 \\ &4&&60&&267&&533&&533&&267&&60&&4 \\ 0&&4&&28&&77&&106&&77&&28&&4&&0\end{array}\right)$$

$$= \begin{array}{*{17}{c}} &&&&&&&&0 \\ &&&&&&&12&&12 \\ &&&&&&84&&180&&84 \\ &&&&&231&&801&&801&&231 \\ &&&&318&&1599&&2586&&1599&&318 \\ &&&231&&1599&&3750&&3750&&1599&&231 \\ &&84&&801&&2586&&3750&&2586&&801&&84 \\ &12&&180&&801&&1599&&1599&&801&&180&&12 \\ 0&&12&&84&&231&&318&&231&&84&&12&&0\end{array}$$

We want to show that the difference between the right hand side and the left hand side is nonnegative, but the difference is

$$\begin{array}{*{17}{c}} &&&&&&&&0 \\ &&&&&&&12&&12 \\ &&&&&&84&&36&&84 \\ &&&&&231&&49&&49&&231 \\ &&&&318&&47&&-422&&47&&318 \\ &&&231&&47&&-778&&-778&&47&&231 \\ &&84&&49&&-422&&-778&&-422&&49&&84 \\ &12&&36&&49&&47&&47&&49&&36&&12 \\ 0&&12&&84&&231&&318&&231&&84&&12&&0\end{array}$$

which is clearly nonnegative by Muirhead's inequality. $\square$

Example 3. Let a, b, c be real numbers such that $a^2+b^2+c^2=1$. Show that $\dfrac{1}{1-ab}+\dfrac{1}{1-bc}+\dfrac{1}{1-ca} \le \dfrac{9}{2}$.

Proof. We use the condition to obtain the equivalent inequality

$$\sum_{cyc} \frac{a^2+b^2+c^2}{a^2+b^2+c^2-ab} \le \frac{9}{2}.$$

Clearing denominators, this is equivalent to

$$2\sum_{cyc}(a^2+b^2+c^2)(a^2+b^2+c^2-ab)(a^2+b^2+c^2-ac) \le 9(a^2+b^2+c^2-ab)(a^2+b^2+c^2-ac)(a^2+b^2+c^2-bc).$$

We write the left hand side as

$$2\sum_{cyc}\begin{pmatrix} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 0 & & 1 \end{pmatrix}\begin{pmatrix} & & 1 & & \\ & -1 & & 0 & \\ 1 & & 0 & & 1 \end{pmatrix}\begin{pmatrix} & & 1 & & \\ & 0 & & -1 & \\ 1 & & 0 & & 1 \end{pmatrix}$$

$$= 2\sum_{cyc}\begin{pmatrix} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 0 & & 1 \end{pmatrix}\begin{pmatrix} & & & & 1 & & & & \\ & & & -1 & & -1 & & & \\ & & 2 & & 1 & & 2 & & \\ & -1 & & -1 & & -1 & & -1 & \\ 1 & & 0 & & 2 & & 0 & & 1 \end{pmatrix}$$

$$= 2\sum_{cyc}\begin{pmatrix} & & & & & & 1 & & & & & & \\ & & & & & -1 & & -1 & & & & & \\ & & & & 3 & & 1 & & 3 & & & & \\ & & & -2 & & -2 & & -2 & & -2 & & & \\ & & 3 & & 1 & & 6 & & 1 & & 3 & & \\ & -1 & & -1 & & -2 & & =2 & & -1 & & -1 & \\ 1 & & 0 & & 3 & & 0 & & 3 & & 0 & & 1 \end{pmatrix}$$

$$= 2\begin{pmatrix} & & & & & & 3 & & & & & & \\ & & & & & -2 & & -2 & & & & & \\ & & & & 9 & & -1 & & 9 & & & & \\ & & & -4 & & -3 & & -3 & & -4 & & & \\ & & 9 & & -3 & & 18 & & -3 & & 9 & & \\ & -2 & & -1 & & -3 & & -3 & & -1 & & -2 & \\ 3 & & -2 & & 9 & & -4 & & 9 & & -2 & & 3 \end{pmatrix}$$

$$= \begin{matrix} & & & & & & 6 & & & & & & \\ & & & & & -4 & & -4 & & & & & \\ & & & & 18 & & -2 & & 18 & & & & \\ & & & -8 & & -6 & & -6 & & -8 & & & \\ & & 18 & & -6 & & 36 & & -6 & & 18 & & \\ & -4 & & -2 & & -6 & & -6 & & -2 & & -4 & \\ 6 & & -4 & & 18 & & -8 & & 18 & & -4 & & 6 \end{matrix}$$

We now write the right hand side as

$$9\left(\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & -1 & & 1 \end{array}\right)\left(\begin{array}{ccccc} & & 1 & & \\ & -1 & & 0 & \\ 1 & & 0 & & 1 \end{array}\right)\left(\begin{array}{ccccc} & & 1 & & \\ & 0 & & -1 & \\ 1 & & 0 & & 1 \end{array}\right)$$

$$= 9\left(\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & -1 & & 1 \end{array}\right)\left(\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & -1 & & -1 & & & \\ & & 2 & & 1 & & 2 & & \\ & -1 & & -1 & & -1 & & -1 & \\ 1 & & 0 & & 2 & & 0 & & 1 \end{array}\right)$$

$$= 9\left(\begin{array}{ccccccccccccc} & & & & & & 1 & & & & & & \\ & & & & & -1 & & -1 & & & & & \\ & & & & 3 & & 0 & & 3 & & & & \\ & & & -2 & & -1 & & -1 & & -2 & & & \\ & & 3 & & -1 & & 5 & & -1 & & 3 & & \\ & -1 & & 0 & & -1 & & -1 & & 0 & & -1 & \\ 1 & & -1 & & 3 & & -2 & & 3 & & -1 & & 1 \end{array}\right)$$

$$= \begin{array}{ccccccccccccc} & & & & & & 9 & & & & & & \\ & & & & & -9 & & -9 & & & & & \\ & & & & 27 & & 0 & & 27 & & & & \\ & & & -18 & & -9 & & -9 & & -18 & & & \\ & & 27 & & -9 & & 45 & & -9 & & 27 & & \\ & -9 & & 0 & & -9 & & -9 & & 0 & & -9 & \\ 9 & & -9 & & 27 & & -18 & & 27 & & -9 & & 9 \end{array}$$

We now want to show that the difference between the right hand side and the left hand side is nonnegative. The difference is

$$\begin{array}{ccccccccccccc} & & & & & & 3 & & & & & & \\ & & & & & -5 & & -5 & & & & & \\ & & & & 9 & & 2 & & 9 & & & & \\ & & & -10 & & -3 & & -3 & & -10 & & & \\ & & 9 & & -3 & & 9 & & -3 & & 9 & & \\ & -5 & & 2 & & -3 & & -3 & & -5 & & & \\ 3 & & -5 & & 9 & & -10 & & 9 & & -5 & & 3 \end{array}$$

Here we have a bit of a sticky situation. Neither AM-GM nor Schur are strong enough to eliminate the $-5a^5b$ terms, so we need to be a bit smarter. Let's look at the expansion of $(a-b)^4$. This is $a^4-4a^3b+6a^2b^2-4ab^3+b^4$. This might be strong enough to help us out here. We have that $\sum_{sym} a^2(a-b)^4 \geq 0$, or

$$\begin{array}{ccccccccccccc} & & & & & & 2 & & & & & & \\ & & & & & -4 & & -4 & & & & & \\ & & & & 7 & & 0 & & 7 & & & & \\ & & & -8 & & 0 & & 0 & & -8 & & & \\ & & 7 & & 0 & & 0 & & 0 & & 7 & & \\ & -4 & & 0 & & 0 & & 0 & & 0 & & -4 & \\ 2 & & -4 & & 7 & & -8 & & 7 & & -4 & & 2 \end{array} \geq 0$$

Subtracting this from the expression above, we see that it suffices to show that the following expression is nonnegative:

$$\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & -1 & & -1 & & & & & \\
 & & & & 2 & & 2 & & 2 & & & & \\
 & & & -2 & & -3 & & -3 & & -2 & & & \\
 & & 2 & & -3 & & 9 & & -3 & & 2 & & \\
 & -1 & & 2 & & -3 & & -3 & & 2 & & -1 & \\
1 & & -1 & & 2 & & -2 & & 2 & & -1 & & 1
\end{array}$$

By AM-GM we have

$$\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & -1 & & -1 & & & & & \\
 & & & & 2 & & 2 & & 2 & & & & \\
 & & & -2 & & -3 & & -3 & & -2 & & & \\
 & & 2 & & -3 & & 9 & & -3 & & 2 & & \\
 & -1 & & 2 & & -3 & & -3 & & 2 & & -1 & \\
1 & & -1 & & 2 & & -2 & & 2 & & -1 & & 1
\end{array}$$

$$\geq \begin{array}{ccccccccccccc}
 & & & & & & 0 & & & & & & \\
 & & & & & 0 & & 0 & & & & & \\
 & & & & \frac{3}{2} & & 2 & & \frac{3}{2} & & & & \\
 & & & -2 & & -3 & & -3 & & -2 & & & \\
 & & \frac{3}{2} & & -3 & & 9 & & -3 & & \frac{3}{2} & & \\
 & 0 & & 2 & & -3 & & -3 & & 2 & & 0 & \\
0 & & 0 & & \frac{3}{2} & & -2 & & \frac{3}{2} & & 0 & & 0
\end{array}$$

$$\geq \begin{array}{ccccccccccccc}
 & & & & & & 0 & & & & & & \\
 & & & & & 0 & & 0 & & & & & \\
 & & & & \frac{1}{2} & & 2 & & \frac{1}{2} & & & & \\
 & & & 0 & & -3 & & -3 & & 0 & & & \\
 & & \frac{1}{2} & & -3 & & 9 & & -3 & & \frac{1}{2} & & \\
 & 0 & & 2 & & -3 & & -3 & & 2 & & 0 & \\
0 & & 0 & & \frac{1}{2} & & 0 & & \frac{1}{2} & & 0 & & 0
\end{array}$$

$$\geq \begin{array}{ccccccccccccc}
 & & & & & & 0 & & & & & & \\
 & & & & & 0 & & 0 & & & & & \\
 & & & & 0 & & 3 & & 0 & & & & \\
 & & & 0 & & -3 & & -3 & & 0 & & & \\
 & & 0 & & -3 & & 9 & & -3 & & 0 & & \\
 & 0 & & 3 & & -3 & & -3 & & 3 & & 0 & \\
0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0
\end{array}$$

which is nonnegative by Schur's inequality. $\square$