Beginner Combinatorics

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Combinatorics is the art of counting. You should already know basic facts about permutations and combinations from your math classes, and there are other basic techniques that I will teach you in this lecture, but preparation for combinatorics problems requires solving problems. As a result, the emphasis of this lecture is placed squarely on solving problems.

1 The Basic Principles

1. How many ways can one arrange a set of n items?

   There are n places for you to put the first element, $n-1$ places for you to put the second, and in general $n-k+1$ places for you to put the kth element. Thus there are a total of $n(n-1)(n-2)\cdots 3\cdot 2\cdot 1 = n!$ possibilities.

2. How many ways can one pick k items out of n items in a particular order?

   Use the same principle as before, but stop before the $k+1$th element. This gives you $n(n-1)(n-2)\cdots(n-k+1) = \frac{n!}{(n-k)!}$ ways.

3. How many ways can one pick k items out of n items if we don't care about the order in which we pick them?

   You've probably learned in math class that the answer to this is $\binom{n}{k} = \frac{n!}{k!(n-k)!}$. However, we can also easily derive this: we use the formula we've derived in the previous problem and realize that there are $k!$ total ordering of the k elements. Thus, there are $\frac{n!}{k!(n-k)!}$ total combinations.

   Why does $\binom{n}{k} = \binom{n}{n-k}$?.

4. How many k-letter words exist in an alphabet with n letters?

   This is just $\overbrace{n\cdot n\cdots n}^{k \text{ times}} = n^k$. Note that we can use this to prove that the number of subsets of a set with n elements is 2^n.

5. Prove Pascal's Identity: $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$.

   We have a set of n items and we want to choose k of them. Mark one item as special. We choose this item to be one of the k chosen items, leaving $\binom{n-1}{k-1}$ ways to choose the others, or we can decide to not choose it, leaving $\binom{n-1}{k}$ ways to choose the others.

2 Examples and New Techniques

1. There are $2n$ players participating in a tennis tournament. How many possible pairings are there for the first round?

   First Solution: We'll use this to illustrate recursion. Let the number of pairings of $2n$ players be P_n. First of all, in the case P_1, there is clearly only 1 way to pair the players. Now suppose we have n players. Choose any player. This player's partner can be chosen in $(2n-1)$ ways. There are now $(n-1)$ pairs to be made. Thus

$$P_1 = 1, P_n = (2n-1)P_{n-1}$$

$$P_n = (2n-1)(2n-3)\cdots 3\cdot 1 = \frac{(2n)!}{2^n n!}$$

Second Solution: Place the $2n$ players in a row in some order; this can be done in $(2n)!$ ways. Now, pair the players in location $(1,2),(3,4),\ldots,(2n-1,2n)$. This can be done in one way. However, our current count includes lots of possible orderings. There are $n!$ different ways we could arrange the pairs and $2\cdot 2\cdots 2 = 2^n$ ways we can permute the elements of each pair. Thus, dividing by $n!\cdot 2^n$ gets us our answer of $\frac{(2n)!}{2^n n!}$.

2. How many diagonals are there in a convex polygon with n sides?

   There are $\binom{n}{2}$ pairs of points, but n of those determine sides. Thus there are $\binom{n}{2} - n = \frac{n(n-3)}{2}$ diagonals.

3. How many intersection points of the diagonals are there?

   This illustrates another key technique - finding a bijection, or one-to-one mapping between two sets. In this case, note that the number of diagonals is equal to the number of quadruples of vertices: $\binom{n}{4}$.

4. Into how many regions is the n-gon divided by the diagonals, if no three diagonals pass through a point?

   First, we have one region (the n-gon). Drawing a diagonal that doesn't intersect any other diagonal adds one region, and drawing a diagonal that intersects k diagonals adds $k+1$ regions. Thus the total number of regions is $1+$ diagonals $+$ intersection points $= 1 + \binom{n}{2} - n + \binom{n}{4}$.

5. How many ways can one travel from the lower left corner to the upper right corner of an m-by-n grid, if you can only move up or to the right?

   There is a bijection between the number of arrangements of the letters in $\overbrace{UU\cdots U}^{m}\overbrace{RR\cdots R}^{n}$, so there are $\binom{m+n}{n} = \binom{m+n}{m}$ paths.

6. (Vladimir Novakovski) Farmer John is lining up his twelve cows in a single line. He has 3 Holsteins, 4 Jerseys, and 5 Guernseys, but he cannot put any two Guernseys next to each other because they tend to fight. How many different arrangements of his cows are there?

   Place the Holsteins and Jerseys first, because they're unrestricted. There are $\binom{4+3}{3}$ ways to place the Holsteins and Jerseys. Now, we can place the Guernseys in the spaces between them or on the ends. There are 6 spaces between Holsteins or Jerseys and 2 at the left and right end, so there are $\binom{8}{5}$ ways to place the Guernseys and a total of $\binom{7}{3}\binom{8}{5} = 1960$ ways to arrange his cows. This problem illustrates the technique of "Stars and Bars", which will be used for the next problem.

7. How many (positive/non-negative/greater than m) solutions exist to the system $a_1+a_2+a_3+\cdots+a_k = n$?

   First, assume all a_i are positive. Draw n circles in a row, and consider drawing $k-1$ lines in the $n-1$ gaps between the circles. Let a_1 be the number of circles before the first line, a_2 be the number of circles between the first and second lines, etc. Clearly, there will be k total a_i (if there are $k-1$ lines), all a_i will be at least 1 (since there is at least 1 circle between each gap) and the sum of the a_i will be n (since each circle is counted exactly once). Thus, there is a bijection between the sequence $a_1,\ldots,a_k$ and the drawing; as the drawing was generated by choosing $k-1$ lines from $n-1$ gaps, there are $\binom{n-1}{k-1}$ drawings, and thus $\binom{n-1}{k-1}$ sequences. For nonnegative a_i, consider the same algorithm, but start with $n-k+1$ circles and cross out $k-1$ of them. You should work this out as an exercise to get the answer $\binom{n+k-1}{k-1}$. Now, extending it to a_i such that $a_i > m$ is easy. Subtract km from each side to get

$$(a_1 - m) + (a_2 - m) + \cdots + (a_k - m) = n - km$$

Each $(a_i - m)$ is a positive integer, so we can apply the previous argument on the sequence $a_1 - m, a_2 - m, \ldots, a_k - m$ to get $\binom{n-km-1}{k-1}$.

3 Essential Knowledge

First, we have the binomial theorem, which you should know from your math classes:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

This gives us another very simple proof that there are 2^n subsets of an n-element set: consider $(1+1)^n = \sum_{k=0}^{n} \binom{n}{k}$. Why are we now done?
Next, we have the principle of inclusion-exclusion. It is, in essence, the statement: "if you count something twice, subtract it once so we only count it once." This is what it looks like for two sets:

$$|A \cup B| = |A| + |B| - |A \cap B|$$

And this is what it looks like for three sets:

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$$

This can be generalized to any number of sets. How would you do that?

With these principles, you can solve just about any combinatorics problem. Combinatorics is much less theoretically dense than most fields, and thus requires more cleverness and intuition, both of which are achieved by doing lots of problems. To that end, here are 7 problems for you.

4 Problems

1. (1) How many ways can you choose an odd number of objects from a set of n objects?
2. (1) You are a cow standing in the upper-left corner of a 4 by 5 grid. Unfortunately, the upper right corner of the grid and the three squares adjacent to it have been overrun with bad-tasting grass. How many ways can you walk to the lower right corner, moving only down or to the right, and avoiding the bad grass?
3. (2) You have 6 cows standing in a line in some order. How many ways can you rearrange them so that no cow is left standing in its initial position?
4. (2) Four more cows have arrived. Your 10 cows now don't care about moving away from their positions. However, they do want to know the number of ways they can rearrange such that they move by at most one position. Please help them.
5. (2) Your 10 cows are now bored of standing in a line, so they've formed a circle. How many ways can the cows rearrange if they may move by at most one position?
6. (2) How many non-negative integer solutions are there to $w + x + y + z \leq 20$?

7. (4) [AIME 1988/15] A secretary has 9 letters to type, which are delivered to her inbox throughout the day. When she has time she takes the letter on top and types it. At noon, letter 8 has already been typed (so letters 1-7 have already been delivered, and letter 9 may or may not have been delivered). How many typing orders are possible for the afternoon?