# Derivation of the Heisenberg Uncertainty Principle

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We start off with our generic wave function $\Psi(x,t)$. This can be written as

$$\Psi(x,t) = A\cos(\omega t - kx)$$

Now, since the angular frequency is defined to be $\omega = 2\pi\nu$ and the wave number is defined to be $k = \frac{2\pi}{\lambda}$, we can substitute these quantities into the equation to get $\Psi(x,t) = A\cos(2\pi\nu t - \frac{2\pi}{\lambda})$. Additionally, we know that $E = h\nu$ and $\lambda = \frac{h}{p}$. Substituting these quantities into the equation, factoring, and replacing $\frac{2\pi}{h}$ with $\frac{1}{\hbar}$, we get an alternate form of the wave function:

$$\Psi(x,t) = A\cos\frac{1}{\hbar}(Et - px)$$

**Theorem 0.1.** (Fourier's Theorems)

1. *Any periodic waveform can be represented by a discrete series of sines and cosines:*

$$f(x) = \sum_{i=1}^{\infty} a_i \cos(\omega_i t - k_i x) + b_i \sin(\omega_i t - k_i x)$$

2. *Any waveform can be represented as a continuous sum of sines and cosines (Fourier transform):*

$$f(x) = \int_{-\infty}^{\infty} A(k)\cos(\omega t - kx) + B(k)\sin(\omega t - kx)\,dx$$

Luckily, for a lot of waveforms, you can get away with just one sine or cosine and thus just one set of coefficients. Also, it turns out that the Fourier conjugate of a very localized waveform will be spread out. Thus, if position and momentum (or energy and time, etc) are Fourier conjugates, and if you know the position to a high degree of accuracy, then you don't know the momentum very well and vice versa. Heisenberg found out that you can quantify this as $\Delta x \Delta p \geq \frac{\hbar}{2}$, and that is what we will derive.

Since $\Psi(x)$ is a waveform, we can write it as a Fourier series:

$$\Psi(x) = \int_{-\infty}^{\infty} g(k) \cos kx \, dk$$

or

$$\Psi(x) = \mathcal{F}[g(k)]$$

How do we find the coefficients? We should expect the classical momentum to be the average value, and other values to be less probable - in other words, some sort of probability distribution. Specifically, the normal distribution, meaning probability $P_k = A_k e^{-(k-k_0)/(2\sigma_k^2)}$. This implies that

$$g(k) = \sqrt{P_k} = \sqrt{A_k} e^{-(k-k_0)/(4\sigma_k^2)}$$

Now, consider the square of the norm of the wavefunction, $|\Psi|^2$. This is a probability distribution; it tells you where the particle is most likely to be. Like earlier, we want the middle value to be the classical position. Like earlier, it happens to be a normal distribution. If we let $x_0$ be the most likely position for the particle, then a normal distribution of the positions is $P_x = A_x e^{-(x-x_0)^2/(2\sigma_x^2)}$.Now, we will denote the envelope of a function $f$ by $f_{\text{env}}$. Since $P_x$ happens to be the upper envelope of $|\Psi(x)|^2$, $\Psi(x)_{\text{env}}$ happens to be

$$\Psi(x)_{\text{env}} = \sqrt{P_x} = \sqrt{A_x} e^{-(x-x_0)^2/(4\sigma_x^2)}$$

Additionally, $\int_{-\infty}^{\infty} P_x \, dx = \int_{-\infty}^{\infty} |\Psi(x)|^2 \, dx = 1$, because the sum of all the probabilities must be 1. This means we can find the coefficients $A_x$ by simply normalizing the normal distribution; this gets us

$$A_x = \frac{1}{\sigma_x \sqrt{2\pi}}$$

Now, we state that $\Psi(x) = \mathcal{F} g(k)$ in the following way:

$$\Psi(x) = \int_{-\infty}^{\infty} g(k) \cos kx \, dk$$

and since $\Psi(x)$ and $g(k)$ are Fourier conjugates the following is true as well:

$$g(k) = \int_{-\infty}^{\infty} \Psi(x) \cos kx \, dx$$

Substituting our results above gives

$$\sqrt{A_k} e^{-(k-k_0)^2/(4\sigma_k^2)} = \int_{-\infty}^{\infty} \frac{1}{\sqrt{\sigma_x \sqrt{2\pi}}} e^{-(x-x_0)^2/(4\sigma_x^2)} \cos kx \, dx \tag{1}$$

Since the function $\Psi(x)\cos kx$ is even, its integral from negative infinity to infinity is simply twice its integral from zero to infinity. The integral of that function is well known and can be found in integral tables:

$$\int_0^\infty e^{-a^2x^2}\cos bx\,dx = \frac{\sqrt{\pi}}{2a}e^{\frac{-b^2}{4a^2}}$$

In the case of $\Psi(x)$, first we note that we may as well choose $x_0 = k_0 = 0$, and we also have $a^2 = \dfrac{1}{4\sigma_x^2} \Leftrightarrow a = \dfrac{1}{2\sigma_x}$ and $b = k$. Let's see how our integral from (1) looks now:

$$\sqrt{A_k}e^{-k^2/(4\sigma_k^2)} = 2\int_0^\infty \frac{1}{\sqrt{\sigma_x}}e^{-x^2/(4\sigma_x^2)}\cos kx\,dx = \frac{2\sqrt{\pi}}{2\left(\frac{1}{2\sigma_x}\right)}e^{-k^2/(4(\frac{1}{4\sigma_x^2}))}$$

The $\sqrt{A_k}$ and $\dfrac{2\sqrt{\pi}}{2\left(\frac{1}{2\sigma_x}\right)}$ are equal (?), so when we cancel that we get

$$e^{\frac{-k^2}{4\sigma_k^2}} = e^{\frac{-k^2}{4\left(\frac{1}{4\sigma_x^2}\right)}}$$

Now we take the natural log of both sides and we're left with

$$\frac{-k^2}{4\sigma_k^2} = \frac{-k^2}{4\left(\frac{1}{4\sigma_x^2}\right)}$$

This simplifies to $\sigma_k^2 = \dfrac{1}{4\sigma_x^2}$ or

$$\sigma_k \cdot \sigma_x = \frac{1}{2}$$

But we know that the wave number $k = \dfrac{p}{\hbar}$, so

$$\sigma_x\sigma_p = \frac{\hbar}{2}$$

Note that this is true only if the probability distribution is normal. If it isn't $\sigma_p\sigma_x$ will be greater, as the normal distribution turns out to have the minimum possible product. This means that we can write our more general physical law as

$$\boxed{\sigma_x\sigma_p \geq \frac{\hbar}{2}}$$

or as

$$\boxed{\Delta x\Delta p \geq \frac{\hbar}{2}}$$